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# PLANE CRACK OF AN ARBITRARY DISCONTINUITY IN A BOUNDED ELASTIC BODY*

R.V. GOL'DSHTEIN and E.I. SHIFRIN

A system of integro-differential equations corresponding to the problem of a plane crack of an arbitrary discontinuity in a bounded elastic body is investigated. It is proved that the integro-differential operator of the system continuously maps $H_{1/2}^{\circ}(G)$ and $H_{-1/2}(G)$ and is a Fredholm operator of index zero. This operator is decomposed into the sum of two operators: one corresponds to the problem of a crack in an unlimited medium, while the other takes account of the influence of the body boundary. When the body boundary is remote from the crack, the second operator is small in the norm, and consequently, the total operator is reversible. This means that the system can be solved by successive approximations. The conditions for convergence of the method depend on the body and crack geometrics and on the material properties. In the sense of the estimates obtained, the crack estimate can be considered remote from the boundary of the body even when it is arbitrarily close to the boundary and has a large diameter but a small area. As an illustration, estimates are calculated for the constants in the condition for convergence of successive approximation for a sphere with a crack in the diametral plane.

1. Let there be a plane crack occupying a domain $G$ of the plane $x_3=0$ in a bounded elastic body $V$. It is assumed that the body surface $S$ is free of forces, while forces $\sigma_{i3}=t_i$, identical in absolute value but opposite in direction, are applied to the crack surfaces. In this case the jumps in the displacements $b_k$ on the crack surfaces satisfy the system of integro-differential equations /1/

$$L_{i3}[\mathbf{b}]-M_{i3}[\mathbf{b}]=t_i,\ i=1,2,3,\ \mathbf{b}=(b_1,b_2,b_3) \tag{1.1}$$

$$M_{i3}[\mathbf{b}]=\int u_k^s(Q)\,S_{ki3}(Q,P)\,dS(Q)-\int t_k^s(Q)\,D_{ki3}(Q,P)\,dS(Q)$$

$$t_j^s=-\sigma_{jm}^s n_m$$

$$\sigma_{ij}^s(P)=-\int b_k(Q)\,S_{kij}(Q,P)\,dS_G(Q),\qquad P\in S$$

$$S_{kij}=-\frac{p\mu}{2\pi R^3}\Big\{3R_{,l}n_l\Big[\delta_{ij}R_{,k}+\frac{\nu}{1-2\nu}(\delta_{ki}R_{,j}+\delta_{kj}R_{,i})-\frac{5}{1-2\nu}R_{,i}R_{,j}R_{,k}\Big]+\frac{3\nu}{1-2\nu}(n_iR_{,j}R_{,k}+n_jR_{,i}R_{,k})+3n_kR_{,i}R_{,j}+n_j\delta_{ki}+n_i\delta_{kj}-\frac{(1-4\nu)}{1-2\nu}n_k\delta_{ij}\Big\}$$

$$D_{kij}=-\frac{p}{4\pi R^2}\Big[\delta_{ki}R_{,j}+\delta_{kj}R_{,i}-\delta_{ij}R_{,k}+\frac{3}{1-2\nu}R_{,i}R_{,j}R_{,k}\Big]$$

$$p=(1-2\nu)/[2(1-\nu)],\quad R=|P-Q|,\quad R_{,l}=(P_l-Q_l)/R$$

Here $t_k^s$ are stresses on the surface $S$ caused by displacement jumps $\mathbf{b}$ in the unbounded body, $u_k^s$ are displacements of the surface $S$ of a body without cracks $V$ if forces $t_k^s$ are applied to $S$, $n_m$ are components of the unit vector, and $\delta_{ij}$ is the Kronecker delta. Here and below the integrals over the volume of the body, its surface and the surface of the crack are different by the signs of the differentials.

The operator $L[\mathbf{b}]$ corresponds to a crack $G$ in an unbounded medium and can be written in the form /1/

$$L[\mathbf{b}]=(L_{13}[\mathbf{b}],L_{23}[\mathbf{b}],L_{33}[\mathbf{b}])-\frac{\mu}{2}F^{-1}\Big\{|\xi|\Big(\delta_{\alpha\beta}+\frac{\nu}{1-\nu}\eta_\alpha\eta_\beta\Big)b_\beta\Big\}=-\frac{\mu}{2}A\mathbf{b}_M,\quad \mathbf{b}_M=(b_1,b_2)$$

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$$\frac{\mu}{2}F^{-1}\{|\xi|\,b_3^{\vee}\} = -\frac{\mu}{2(1-\nu)}\,p_G\Lambda b_3$$
$$\alpha = 1, 2;\quad \beta = 1, 2;\quad \eta_\alpha = \xi_\alpha/|\xi|;\quad x = (x_1, x_2) \in G;$$
$$b_i^{\vee} = \iint\limits_{-\infty}^{\infty} b_i(x)\,e^{i(x,\xi)}\,dx;\qquad (x, \xi) = x_1\xi_1 + x_2\xi_2$$

The displacements $u_k^s$ in the expression for $M_{i3}[\mathbf{b}]$ are not expressed explicitly in terms of $t_k^s$, consequently it is impossible to write down a dependence of $u_k^s$ on $\mathbf{b}$. However, the displacements $u_k^s$ can be obtained as the solutions of elasticity theory equations for a body $V$ without cracks for stresses $t_k^s$ given on the surface $S$. The single-valued solvability of these equations to the accuracy of a rigid displacement is proved (see /2/, for instance), where for $t_k^s \in L_2(S)$ the displacements lie in $H_1(V)$, and therefore, $u_k^s \in H_{1/2}(S)$. An estimate of the $L_2$-norm of $u_k^s$ is presented below.

2. Let us show that the operator $L[\mathbf{b}] - M[\mathbf{b}]$, $M[\mathbf{b}] = (M_{13}[\mathbf{b}], M_{23}[\mathbf{b}], M_{33}[\mathbf{b}])$ maps $H_{1/2}^{\circ}(G)$ continuously into $H_{-1/2}(G)$, where it is a Fredholm operator of index zero.

We recall that for vector-function spaces $H_s^{\circ}(G)$ and $H_s(G)$ are defined analogously to the case of scalar functions

$$\|\mathbf{b}\|_{1/2}^2 = \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty}(1+|\xi|)\,|\mathbf{b}^{\vee}(\xi)|^2\,d\xi$$
$$|\mathbf{b}^{\vee}(\xi)|^2 = |b_1^{\vee}(\xi)|^2 + |b_2^{\vee}(\xi)|^2 + |b_3^{\vee}(\xi)|^2$$

$\mathbf{t} \in H_{-1/2}(G)$ if and only if $t_i \in H_{-1/2}(G)$, $i = 1, 2, 3$

$$\|\mathbf{t}\|_{-1/2}^2 = \inf_{l\mathbf{t}}\frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty}\frac{|l\mathbf{t}^{\vee}(\xi)|^2}{1+|\xi|}\,d\xi$$

where $l\mathbf{t}$ is the continuation of $\mathbf{t}$ in $H_{-1/2}(R^2)$, $l\mathbf{t} = (l_1t_1, l_2t_2, l_3t_3)$, $l_it_i$ is the continuation of $t_i$ in $H_{-1/2}(R^2)$.

That $L[\mathbf{b}] - M[\mathbf{b}]$ is a Fredholm operator will follow from the reversibility of the operator $L[\mathbf{b}]$, and the complete continuity of the operator $M[\mathbf{b}]$. Moreover, if the body boundary is sufficiently remote from the crack surface, then the norm of operator $M[\mathbf{b}]$ is small, consequently, the operator $L[\mathbf{b}] - M[\mathbf{b}]$ is reversible, and the system (1.1) can be solved by successive approximations.

Now $L[\mathbf{b}]$ is a first order linear operator, and hence maps $H_{1/2}^{\circ}(G)$ continuously into the conjugate space $H_{-1/2}(G)$. Therefore, it is sufficient to prove the coercivity $|(L[\mathbf{b}], \mathbf{b})| \geqslant \mathrm{const}\,\|b\|_{1/2}^2$ for the reversibility of the operator $L[\mathbf{b}]$. The coercivity of $L[\mathbf{b}]$ follows from the results in /3,4/. In conformity with /3/

$$\left|-\frac{\mu}{2(1-\nu)}(\Lambda b_3, b_3)\right| \geqslant \frac{\mu}{2(1-\nu)}\frac{\sqrt{\pi}\lambda_1(K_1)}{[\mu(G)]^{1/2}}\|b_3\|^2 \tag{2.1}$$

where $\mu(G)$ is the area of the domain $G$, $\|\cdot\|$ is the $L_2$-norm, $\lambda_1(K_1)$ is the minimal eigennumber of the operator $p_{K_1}\Lambda$ /3/, and $K_1$ is a unit circle. It follows from (2.1)

$$(\Lambda b_3, b_3) \geqslant \varkappa\|b_3\|_{1/2}^2;\ \varkappa = \sqrt{\pi}\lambda_1(K_1)\{[\mu(G)]^{1/2} + \sqrt{\pi}\lambda_1(K_1)\}^{-1}$$
$$\left|-\frac{\mu}{2(1-\nu)}(\Lambda b_3, b_3)\right| \geqslant \frac{\mu\varkappa}{2(1-\nu)}\|b_3\|_{1/2}^2 \tag{2.2}$$

In conformity with /4/

$$\left|-\frac{\mu}{2}(A\mathbf{b}_M, \mathbf{b}_M)\right| \geqslant \frac{\mu}{2}\frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty}|\xi|\,|\mathbf{b}_M^{\vee}(\xi)|^2\,d\xi \tag{2.3}$$

Using (2.2) and (2.3) we obtain

$$\left|-\frac{\mu}{2}(A\mathbf{b}_M, \mathbf{b}_M)\right| \geqslant \frac{\mu\varkappa}{2}\|\mathbf{b}_M\|_{1/2}^2 \tag{2.4}$$

From the inequalities (2.2) and (2.4) it follows

$$|(L[\mathbf{b}], \mathbf{b})| \geqslant \frac{\mu\varkappa}{2}\|\mathbf{b}\|_{1/2}^2 \tag{2.5}$$

Therefore, the coercivity, as well as the reversibility, of the operator $L[\mathbf{b}]$ are proved.

To prove the complete continuity of the operator $M[\mathbf{b}]$ it is sufficient to prove the inequality

$$\| M[\mathbf{b}] \| \leqslant \text{const} \| \mathbf{b} \|_{1/2} \tag{2.6}$$

Indeed, from the inequality (2.6) there follows $M$: $H_{1/2}^{\circ}(G) \to H_0(G)$ is a continuous mapping, and since $i$: $H_0(G) \to H_{-1/2}(G)$ is a completely continuous imbedding, then $M$: $H_{1/2}^{\circ}(G) \to H_{-1/2}(G)$ is a completely continuous operator.

To obtain an estimate of the norm of $M[\mathbf{b}]$ we first estimate the norm of $t_k^s$ in terms of $\| \mathbf{b} \|_{1/2}$

$$[t_i^s]^2 = \int (t_i^s)^2 \, dS = \int (\sigma_{ij}^s n_j)^2 \, dS \leqslant \int \Big| \sum_j (\sigma_{ij}^s)^2 \Big\| \sum_j n_j^2 \Big| \, dS = \int \Big| \sum_j (\sigma_{ij}^s)^2 \Big| \, dS = [\sigma_i^s]^2$$

$$| \sigma_{ij}^s(P) | = \Big| \int b_k(Q) S_{kij}(Q, P) \, dS_G(Q) \Big| \leqslant \| \mathbf{b} \| \Big( \int \sum_k S_{kij}^2(Q, P) \, dS_G(Q) \Big)^{1/2} \tag{2.7}$$

$$\| \mathbf{b} \|^2 = \gamma^2 \| \mathbf{b} \|^2 + (1 - \gamma^2) \| \mathbf{b} \|^2 \leqslant \gamma^2 \| \mathbf{b} \|^2 + (1 - \gamma^2) [\mu(G)]^{1/2} (\Lambda \mathbf{b}, \mathbf{b}) [\sqrt{\pi} \lambda_1(K_1)]^{-1}$$

We select $\gamma^2 = [\mu(G)]^{1/2} \{[\mu(G)]^{1/2} + \sqrt{\pi}\lambda_1(K_1)\}^{-1}$, then $\gamma^2 = (1 - \gamma^2) [\mu(G)]^{1/2} [\sqrt{\pi}\lambda_1(K_1)]^{-1}$. We hence obtain

$$\| \mathbf{b} \| \leqslant \gamma \| \mathbf{b} \|_{1/2} \tag{2.8}$$

From the inequalities (2.7) and (2.8) it follows

$$| \sigma_{ij}^s(P) | \leqslant \gamma \| \mathbf{b} \|_{1/2} \Big( \int \sum_k S_{kij}^2(Q, P) \, dS_G(Q) \Big)^{1/2}$$

$$[t_i^s] \leqslant \Big( \int \sum_j [\sigma_{ij}^s(P)]^2 \, dS(P) \Big)^{1/2} \leqslant$$

$$\gamma \| \mathbf{b} \|_{1/2} \Big[ \int \Big( \int \sum_{k,j} S_{kij}^2(Q, P) \, dS_G(Q) \Big) dS(P) \Big]^{1/2} \leqslant$$

$$\gamma \| \mathbf{b} \|_{1/2} \Big[ \int \Big( \int \sum_{k,j} S_{kij}^2(Q, P) \, dS(P) \Big) dS_G(Q) \Big]^{1/2}$$

$$[\mathbf{t}^s] = \Big( \sum_i [t_i^s]^2 \Big)^{1/2} \leqslant \gamma \| \mathbf{b} \|_{1/2} \Big[ \int \Big( \int \sum_{k,i,j} S_{kij}^2(Q, P) \, dS(P) \Big) dS_G(Q) \Big]^{1/2} \leqslant \tag{2.9}$$

$$\gamma \| \mathbf{b} \|_{1/2} [S]_G [\mu(G)]^{1/2}$$

$$[S]_G = \sup \int \sum_{k,i,j} S_{kij}^2(Q, P) \, dS(P)$$

Here and below the sup is taken in $Q \in G$. The index of $G$ indicates that the normals are taken at the point $Q$ to the domain $G$ in the expressions for $S_{kij}(Q, P)$. From the expressions for $M_{i3}[\mathbf{b}]$ we obtain

$$| M_{i3}[\mathbf{b}] | \leqslant [\mathbf{u}^s] \Big( \int \sum_k S_{ki3}^2(Q, P) \, dS(P) \Big)^{1/2} + [\mathbf{t}^s] \Big( \int \sum_k D_{ki3}^2(Q, P) \, dS(P) \Big)^{1/2} \tag{2.10}$$

$$[\mathbf{u}^s]^2 = \int \sum_i [u_i^s(P)]^2 \, dS(P)$$

We estimate the $L_2$-norm of $M[\mathbf{b}]$ by using (2.10)

$$\| M[\mathbf{b}] \| = \Big( \int \sum_i M_{i3}^2[\mathbf{b}] \, dS_G \Big)^{1/2} \leqslant ([\mathbf{u}^s][S_3]_s + [\mathbf{t}^s][D_3]_s) [\mu(G)]^{1/2} \tag{2.11}$$

$$[S_3]_s^2 = \sup \int \sum_{k,i} S_{ki3}^2(Q, P) \, dS(P)$$

where the normals to the surface $S$ at the point $P$ are taken in the expression for $S_{ki3}(Q, P)$

$$[D_3]_s^2 = \sup \int \sum_{k,i} D_{ki3}^2(Q, P) \, dS(P)$$

To esimate $\| M[\mathbf{b}] \|$ it remains to estimate $[\mathbf{u}^s]$. As already noted, $u_k^s(P)$ is the shift of the elasticity of a body $V$ without cracks on the boundary $S$ for loads $t_i^s$ given on $S$. There is arbitrariness in the determination of $u_k^s$ since $u_k^s$ is determined to the accuracy of a rigid displacement. Integrals including $u_k^s$ and in the expression for $M_{i3}[\mathbf{b}]$ are independent of the specific selection of $u_k^s$, we shall consider that

$$\int u_k \, dv = 0, \quad k = 1, 2, 3 \tag{2.12}$$

$$\int w_{ij}\,dv = 0, \quad i, j = 1, 2, 3, \quad w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

where $u_i$ are the displacements in the body $V$. Under the conditions (2.12), the $[\mathbf{u}^s]$ can be estimated. Let $e_{ij}$ be the strain tensor, $E_{ij}$ is the stress tensor in the body $V$ without cracks with the load $\mathbf{t}^s$ on the surface:

$$\int e_{ij}E_{ij}\,dv = \int \frac{\partial u_i}{\partial x_j} E_{ij}\,dv = \int \frac{\partial}{\partial x_j}(u_i E_{ij})\,dv -$$

$$\int u_i \frac{\partial E_{ij}}{\partial x_j}\,dv = \int u_i^s E_{ij}\cos(n, x_j)\,dS = \int u_i^s t_i^s\,dS \leqslant [\mathbf{u}^s]\,[\mathbf{t}^s]$$

The displacements $u_k$ satisfy the conditions (2.12), hence, the second Korn inequality holds

$$\int e_{ij}E_{ij}\,dv \geqslant C_k \int \sum_i |\operatorname{grad} u_i|^2\,dv$$

Therefore

$$C_k \int \sum_i |\operatorname{grad} u_i|^2\,dv \leqslant [\mathbf{u}^s]\,[\mathbf{t}^s] \tag{2.13}$$

By the theorem about traces /2,5/

$$[\mathbf{u}^s]^2 \leqslant C_{sp} \int \sum_i |\operatorname{grad} u_i|^2\,dv \tag{2.14}$$

From (2.13) and (2.14)

$$\left(\int \sum_i |\operatorname{grad} u_i|^2\,dv\right)^{1/2} \leqslant C_{sp}^{1/2} C_k^{-1}\,[\mathbf{t}^s]$$

$$[\mathbf{u}^s] \leqslant C_{sp}C_k^{-1}\,[\mathbf{t}^s] \tag{2.15}$$

Substituting the inequality (2.15) into (2.11), we obtain

$$\|M\,[\mathbf{b}]\| \leqslant [\mathbf{t}^s]\,B\,[\mu\,(G)]^{1/2} \tag{2.16}$$

$$B = [C_{sp}^2 C_k^{-2}\,[S_3]_s^2 + [D_3]_s^2 + 2C_{sp}C_k^{-1}\,[S_3]_s\,[D_3]_s]^{1/2}$$

There follows from (2.9) and (2.16)

$$\|M\,[\mathbf{b}]\| \leqslant \gamma\,\|\mathbf{b}\|_{1/2}\,[S]_G\,[\mu\,(G)]\,B \tag{2.17}$$

Therefore, the inequality (2.6) is set.

3. To prove the reversibility of the operator $L\,[\mathbf{b}] - M\,[\mathbf{b}]$ in the case when the body boundary is remote from the crack surface, we replace the system (1.1) by an equivalent system

$$\mathbf{b} - R_\infty M\,[\mathbf{b}] = \mathbf{q}; \quad \mathbf{b} \in H_{1/2}^\circ(G), \quad \mathbf{q} = R_\infty \mathbf{t} \in H_{1/2}^\circ(G) \tag{3.1}$$

$R_\infty\colon H_{-1/2}(G) \to H_{1/2}^\circ(G)$ is the inverse operator to $L\,[\mathbf{b}]$ (such an operator exists according to the proof above).

Let $\Omega\mathbf{b}$ denote the operator mapping $\mathbf{b}$ into $\mathbf{q} + R_\infty M\,[\mathbf{b}]$, where $\Omega\colon H_{1/2}^\circ(G) \to H_{1/2}^\circ(G)$. The system (3.1) takes the form

$$\Omega\mathbf{b} = \mathbf{b}, \quad \mathbf{b} \in H_{1/2}^\circ(G)$$

For a sufficiently remote boundary of the body from the crack surface, the operator $\Omega$ is compressive, i.e.,

$$\|\Omega\mathbf{b}^1 - \Omega\mathbf{b}^2\|_{1/2} \leqslant \theta\,\|\mathbf{b}^1 - \mathbf{b}^2\|_{1/2} = \theta\,\|\mathbf{d}\|_{1/2} \tag{3.2}$$

$$0 < \theta < 1, \quad \mathbf{d} = \mathbf{b}^1 - \mathbf{b}^2, \quad \forall \mathbf{b}^1, \mathbf{b}^2$$

Indeed

$$\|\Omega\mathbf{b}^1 - \Omega\mathbf{b}^2\|_{1/2} = \|\mathbf{q} + R_\infty M\,[\mathbf{b}^1] - \mathbf{q} - R_\infty M\,[\mathbf{b}^2]\|_{1/2} = \|R_\infty M\,[\mathbf{d}]\|_{1/2}. \tag{3.3}$$

Let $L\,[\mathbf{b}] = \mathbf{f}$, where $\mathbf{f} \in L_2\,(G)$, then

$$|(L\,[\mathbf{b}], \mathbf{b})| \leqslant \|\mathbf{f}\|\,\|\mathbf{b}\| \leqslant \gamma\,\|\mathbf{f}\|\,\|\mathbf{b}\|_{1/2} \tag{3.4}$$

From (2.5) and (3.4) we have $\mu\varkappa\,\|\mathbf{b}\|_{1/2}^2 2^{-1} \leqslant \gamma\,\|\mathbf{f}\|\,\|\mathbf{b}\|_{1/2}$, or

$$\|\mathbf{b}\|_{1/2} \leqslant 2\gamma\,\|\mathbf{f}\|\,\mu^{-1}\,\varkappa^{-1} \tag{3.5}$$

We conclude from (3.3), (3.5) and (217) that

$$\|\Omega\mathbf{b}^1 - \Omega\mathbf{b}^2\|_{1/2} \leqslant 2\gamma\,\|M\,[\mathbf{d}]\|\,\mu^{-1}\,\varkappa^{-1} \leqslant 2\gamma^2\mu^{-1}\,\varkappa^{-1}\,\|\mathbf{d}\|_{1/2}\,[\mu\,(G)]B$$

Let us note that $\gamma^2\varkappa^{-1} = [\mu(G)]^{1/2}\pi^{-1/2}\lambda_1^{-1}(K_1)$, therefore

$$\| \Omega \mathbf{b}^1 - \Omega \mathbf{b}^2 \|_{1/2} \leqslant 2 [\mu(G)]^{3/2}\mu^{-1}\pi^{-1/2}\lambda_1^{-1}(K_1) \| \mathbf{d} \|_{1/2} [S]_G B \tag{3.6}$$

Because of (3.6), we can take as $\theta$ in (3.2)

$$\theta = 2 [\mu(G)]^{3/2}\mu^{-1}\pi^{-1/2}\lambda_1^{-1}(K_1)[S]_G B \tag{3.7}$$

If the crack diminishes in the body $V$, then the quantities $C_{sp}$ and $C_k$ do not change since they are independent of the crack, the quantities $[S]_G$, $[S_3]_s$, $[D_3]_s$ do not increase, and $[\mu(G)]^{3/2}$ diminishes. Therefore, by decreasing the crack it is always possible to achieve that the quantity $\theta$ does not exceed unity and the operator $\Omega$ is compressive. We note that in the sense of (3.7) the body boundary can be remote from the crack surface even in the case when the crack is arbitrarily close to the boundary, has a large diameter, but sufficiently small area, which assures satisfaction of the condition of compressibility of the operator $\Omega$. We later calculate a specific estimate of the quantity $\theta$ for the case of a sphere, which results from (3.7).

4. If it is taken into account that $\lambda_1(K_1) \approx 2$ /3/, then we obtain in place of (3.7)

$$\theta = [\mu(G)]^{3/2}\mu^{-1}\pi^{-1/2}[S]_G B \tag{4.1}$$

It can be established as a result of awkward calculations that

$$[D_3]_s = \sup \left\{ \int \frac{[2 - 8\nu + 8\nu^2 + (16 - 16\nu + 4\nu^2) R^2_{,3}]}{64\pi^2 R^4 (1-\nu)^2} dS \right\}^{1/2} = \sup \left\{ \int \frac{[\beta_1 + \beta_2 R^2_{,3}]}{64\pi^2 R^4 (1-\nu)^2} dS \right\}^{1/2}$$

$$[D_3]_s \leqslant \frac{1}{8\pi(1-\nu)} \sup \left( \int R^{-8} dS \right)^{1/4} \sup \left[ \int (\beta_1^2 + 2\beta_1\beta_2 R^2_{,3} + \beta_2^2 R^4_{,3}) dS \right]^{1/4} \tag{4.2}$$

$$[S]_G = \sup \left\{ \int \frac{p^2\mu^2}{4\pi^2 R^6 (1-2\nu)^2} [10 - 20\nu + 30\nu^2 + (66 + 12\nu - 18\nu^2) R^2_{,3}] dS \right]^{1/2} =$$

$$\frac{p\mu}{2\pi(1-2\nu)} \sup \left\{ \int R^{-6} (\alpha_1 + \alpha_2 R^2_{,3}) dS \right\}^{1/2} = \frac{\mu}{4\pi(1-\nu)} \sup \left\{ \int R^{-6} (\alpha_1 + \alpha_2 R^2_{,3}) dS \right\}^{1/2}$$

It hence follows that

$$[S]_G \leqslant \frac{\mu}{4\pi(1-\nu)} \sup \left( \int R^{-12} dS \right)^{1/4} \sup \left[ \int (\alpha_1^2 + 2\alpha_1\alpha_2 R^2_{,3} + \alpha_2^2 R^4_{,3}) dS \right]^{1/4} \tag{4.3}$$

$$[S_3]_s = \sup \left\{ \int \frac{p^2\mu^2}{4\pi^2 R^6 (1-2\nu)^2} [5 - 6\nu + 5\nu^2 + (69 + 54\nu - 15\nu^2) R^2_{,3} +$$

$$(4 + 28\nu - 24\nu^2) n_3^2] dS \right\}^{1/2} = \frac{\mu}{4\pi(1-\nu)} \sup \left\{ \int R^{-6} (\gamma_1 + \gamma_2 R^2_{,3} + \gamma_3 n_3^2) dS \right\}^{1/2}$$

From the expression for $[S_3]_s$ we obtain the estimate

$$[S_3]_s \leqslant \frac{\mu}{4\pi(1-\nu)} \sup \left( \int R^{-12} dS \right)^{1/4} \sup \left[ \int (\gamma_1^2 + \gamma_2^2 R^4_{,3} + \gamma_3^2 n_3^4 + 2\gamma_1\gamma_2 R^2_{,3} + 2\gamma_1\gamma_3 n_3^3 + 2\gamma_2\gamma_3 R^2_{,3} n_3^2) dS \right]^{1/4} \tag{4.4}$$

To estimate the constants $C_{sp}, C_k$ the form of the domain $V$ must be known. Also the shape and location of the crack $G$ must still be given for a further estimate of $[D_3]_s, [S]_G, [S_3]_s$.

5. Let $V$ be a sphere of radius $R$. In this case the quantity $C_k$ is calculated exactly in /6/. In particular, $C_k = \mu/2$ for $\nu \geqslant 1/14$. Let us calculate $C_{sp}$. We turn to the spherical coordinates $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r\cos\theta$.

Let us consider the sufficiently smooth function $F(x, y, z) = f(r, \theta, \varphi)$ such that

$$\int F(x, y, z) \, dv = 0 \tag{5.1}$$

Let us expand $f(r, \theta, \varphi)$ in a series of spherical functions

$$f(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{q=1}^{n_q} a_{nq}(r) \; Y_n^{(q)}(\theta, \varphi) \tag{5.2}$$

Here $Y_n^{(q)}(\theta, \varphi)$ is the $q$-th spherical function corresponding to the $n$-th eigennumber of the operator $\delta$:

$$\delta g(\theta, \varphi) = -\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial g}{\partial\theta}\right) + \frac{1}{\sin\theta}\,\frac{\partial^2 g}{\partial\varphi^2}\right]$$

We consider the normalized system of spherical functions to be chosen

$$F\,|_s = f\,|_s = \sum_{n=0}^{\infty}\sum_{q=1}^{n_q} a_{nq}(R)\, Y_n^{(q)}(\theta, \varphi) \tag{5.3}$$

It follows from (5.3) that

$$\int F^2\, dS = \sum_{n=0}^{\infty}\sum_{q=1}^{n_q} R^2 a_{nq}^2(R) \tag{5.4}$$

Since

$$\int |\operatorname{grad} F|^2\, dv = \int_0^R r^2 \int\left[\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial\varphi}\right)^2\right] dS_1\, dr$$

where $S_1$ is a unit sphere, we have

$$\int |\operatorname{grad} F|^2\, dv = \int_0^R r^2 \int\Big[\Big(\sum_{n=0}^{\infty}\sum_{q=1}^{n_q} a'_{nq}(r)\, Y_n^{(q)}(\theta, \varphi)\Big) \times \tag{5.5}$$

$$\Big(\sum_{p=0}^{\infty}\sum_{m=1}^{p_m} a'_{pm}(r)\, Y_p^{(m)}(\theta, \varphi)\Big) + \frac{1}{r^2}\Big(\sum_{n=0}^{\infty}\sum_{q=1}^{n_q} a_{nq}(r)\,\frac{\partial Y_n^{(q)}(\theta, \varphi)}{\partial\theta}\Big)\frac{\partial f}{\partial\theta} +$$

$$\frac{1}{r^2\sin^2\theta}\Big(\sum_{n=0}^{\infty}\sum_{q=1}^{n_q} a_{nq}(r)\,\frac{\partial Y_n^{(q)}(\theta, \varphi)}{\partial\varphi}\Big)\frac{\partial f}{\partial\varphi}\Big]\, dS_1\, dr =$$

$$\int_0^R r^2\Big[\sum_{n=0}^{\infty}\sum_{q=1}^{n_q} (a'_{nq}(r))^2 + \frac{1}{r^2}\sum_{n=1}^{\infty}\sum_{q=1}^{n_q} a_{nq}(r)\int\Big(\frac{\partial Y_n^{(q)}(\theta, \varphi)}{\partial\theta}\,\frac{\partial f}{\partial\theta} +$$

$$\frac{1}{\sin^2\theta}\,\frac{\partial Y_n^{(q)}(\theta, \varphi)}{\partial\varphi}\,\frac{\partial f}{\partial\varphi}\Big) dS_1\Big]\, dr =$$

$$\int_0^R r^2\Big[\sum_{n=0}^{\infty}\sum_{q=1}^{n_q} (a'_{nq}(r))^2 + r^{-2} n(n+1)\, a_{nq}^2(r)\Big]\, dr$$

Because of (5.1) and (5.2) and the fact that for $n \geqslant 1$

$$\int Y_n^{(q)}(\theta, \varphi)\, dS_1 = 0$$

we have

$$\int_0^R r^2 a_{01}(r)\, dr = 0 \tag{5.6}$$

It follows from (5.4) and (5.5) that the best constants $D_1$ and $D_2$ must be found such that

$$R^2 a_{01}^2(R) \leqslant D_1 \int_0^R r^2 (a'_{01}(r))^2\, dr \tag{5.7}$$

for functions $a_{01}$ bounded and satisfying (5.6) and

$$R^2 a_{nq}^2(R) \leqslant D_2 \int_0^R [r^2 (a'_{nq}(r))^2 + n(n+1)\, a_{nq}^2(r)]\, dr \tag{5.8}$$

for bounded functions $a_{nq}(r)$.

We note that the best $D_2$ is obtained for $n = 1$. Hence, instead of (5.8) it is sufficient to consider just the inequality

$$R^2 a^2(R) \leqslant D_2 \int_0^R [r^2 (a'(r))^2 + 2a^2(r)]\, dr$$

$$D_2^{-1} = \inf_{a(r)} \int_0^R [r^2 (a'(r))^2 + 2a^2(r)]\, dr\, (R^{-2} a^{-2}(R))$$

Without limiting the generality, it can be considered that $a(R) = 1$. Therefore

$$D_2^{-1} = \inf_{a(r),\, a(R)=1} \int_0^R [r^2 (a'(r))^2 + 2a^2(r)]\, dr\, (R^{-2}) \tag{5.9}$$

We assume that the minimum is achieved in (5.9) for $a(r) = g(r)$, $h(r)$ is a bounded function and $h(R) = 0$

$$\int_0^R [r^2(g'(r)+h'(r))^2 + 2(g(r)+h(r))^2]\,dr = \int_0^R [r^2(g'(r))^2 + 2g^2(r)]\,dr + \int_0^R [r^2(h'(r))^2 + 2h^2(r)]\,dr + 2\int_0^R [r^2g'(r)h'(r) + 2g(r)h(r)]\,dr$$

A minimum is realized on $g(r)$, hence, for any allowable function $h(r)$

$$\int_0^R [r^2g'(r)h'(r) + 2g(r)h(r)]\,dr = -\int_0^R h(r)[r^2g''(r) + 2rg'(r) - 2g(r)]dr = 0$$

Therefore

$$r^2g''(r) + 2rg'(r) - 2g(r) = 0 \tag{5.10}$$

Solving (5.10), we obtain $g(r) = C_1/r^2 + C_2r$. Since $g(r)$ is a bounded function, $C_1 = 0$. Therefore, $g(r) = C_2r$ and since $g(R) = 1$, then $g(r) = rR^{-1}$

$$D_2^{-1} = R^{-2}\int_0^R [r^2R^{-2} + 2r^2R^{-2}]\,dr = R^{-1}$$

Therefore, it is established that $D_2 = R$. We now determine $D_1$. We rewrite the inequality (5.7) in the form

$$D_1^{-1} = \inf_{a_{01}(r)} \left[\int_0^R r^2(a'_{01}(r))^2dr\right](R^{-2}a_{01}^{-2}(R))$$

Without limiting the generality it can be assumed that $a_{01}(R) = 1$. Then

$$D_1^{-1} = \inf_{a_{01}(r),\ a_{01}(R)=1} \int_0^R r^2(a'_{01}(r))^2\,dr\,(R^{-2}) \tag{5.11}$$

For $a_{01}(r) = g(r)$ let the minimum of (5.11) be achieved, $h(r)$ is a bounded function, $h(R) = 0$, and $h(r)$ satisfies condition (5.6)

$$\int_0^R r^2(g'(r)+h'(r))^2\,dr = \int_0^R r^2(g'(r))^2dr + \int_0^R r^2(h'(r))^2\,dr + 2\int_0^R r^2g'(r)h'(r)\,dr$$

Since the minimum is realized on $g(r)$

$$\int_0^R r^2g'(r)h'(r)\,dr = -\int_0^R h(r)[2rg'(r) + r^2g''(r)]\,dr = 0 \tag{5.12}$$

It follows from (5.12) and condition (5.6) for $h(r)$

$$r^2g''(r) + 2rg'(r) = Cr^2, \quad C = \text{const} \tag{5.13}$$

Solving (5.13), we obtain $g(r) = C_1r^{-1} + C_2 + Cr^26^{-1}$. Since $g(r)$ is a bounded function, then $C_1 = 0$. Since $g(R) = 1$

$$C_2 + CR^26^{-1} = 1 \tag{5.14}$$

Because $g(r)$ satisfies (5.6)

$$\int_0^R r^2g(r)\,dr = \int_0^R [C_2r^2 + Cr^46^{-1}]\,dr = C_2R^33^{-1} + CR^5(30)^{-1} = 0$$

or

$$C_2 + 0.1CR^2 = 0 \tag{5.15}$$

Solving (5.14) and (5.15), we determine $C = 15R^{-2}$; $C_2 = -1.5$. Therefore

$$g(r) = -1.5 + 15r^2R^{-2}6^{-1}; \quad g'(r) = 5rR^{-2}$$

$$D_1^{-1} = R^{-2}\int_0^R 25r^4R^{-4}\,dr = 5R^{-1}; \quad D_1 = 0.2R$$

In connection with the fact that $D_2 > D_1$, for a sphere $C_{sp} = D_2 = R$.

**6.** To estimate the integrals in the estimates (4.2), (4.3), (4.4), it is necessary to calculate

$$H_n = \int_S n_3^{2n}dS, \quad I_n = \int_S R_{,3}^{2n}dS; \quad n = 1,2$$

$$J_1 = \int\limits_S R^{-8}\,dS, \quad J_2 = \int\limits_S R^{-12}\,dS, \quad K = \int\limits_S R^2_{,3} n_3^2 dS$$

For definiteness, we assume $V$ to be a sphere of unit radius. Then

$$H_1 = \int\limits_0^{\pi}\int\limits_0^{2\pi} \cos^2\theta \sin\theta\, d\varphi\, d\theta = \frac{4\pi}{3}$$

$$H_2 = \int\limits_0^{\pi}\int\limits_0^{2\pi} \cos^4\theta \sin\theta\, d\varphi\, d\theta = \frac{4\pi}{5}$$

The integrals $J_1, J_2$ evidently depend on just the distance $\rho$ between the point $Q$ and the center of the sphere. We assume the point $Q$ to be on the axis $z$. The center of the sphere coincides with the origin. Then $Q = (0, 0, \rho)$ and $P = (x, y, z)$.

In the spherical coordinate system

$$J_1(\rho) = \int\limits_0^{\pi}\int\limits_0^{2\pi} \frac{\sin\theta\, d\varphi\, d\theta}{R^8} = \frac{4\pi}{3}\,\frac{(3+\rho^2)(1+3\rho^2)}{(1-\rho)^6(1+\rho)^6}$$

$$J_2(\rho) = \int\limits_0^{\pi}\int\limits_0^{2\pi} \frac{\sin\theta\, d\varphi\, d\theta}{R^{12}} = \frac{4\pi}{10}\,\frac{(5+10\rho^2+\rho^4)[(1+\rho)^5+(1-\rho)^5]}{(1-\rho)^{10}(1+\rho)^{10}}$$

$$R^2 = (1 + \rho^2 - 2\rho\cos\theta)$$

$J_1(\rho)$, $J_2(\rho)$ are increasing functions of $\rho$. To estimate the quantities $I_n$ and $K$ it is necessary to have more exact data about the crack $G$. It is later assumed that $G$ lies in the diametral plane of the sphere $z = 0$. Since $Q$ lies in the plane $z = 0$, then by a change of variable $I_n$ can be transformed in such a way that it would agree with the integral that is obtained in the case when $Q$ lies on the $z$ axis and has the coordinates $(0, 0, \rho)$, but a projection on the $x$-axis is taken for the vector $P - Q$. Therefore

$$I_1(\rho) = \int \frac{x^2}{R^2} dS_1 = \int\limits_0^{\pi}\int\limits_0^{2\pi} \frac{\sin^2\theta\cos^2\varphi\sin\theta}{R^2} d\varphi\, d\theta = \frac{\pi}{8\rho^3}\{4\rho(1+\rho^2) - 2(1-\rho)^2 \times [\ln(1+\rho) - \ln(1-\rho)]\}$$

It can be seen that the function decreases as $\rho$ grows, and hence

$$I_1(\rho) \leqslant I_1(0) = H_1 = 4\pi/3$$

$$I_2(\rho) = \int \frac{x^4}{R^4}\, dS_1 = \int\limits_0^{\pi}\int\limits_0^{2\pi} \frac{\sin^4\theta\cos^4\varphi\sin\theta}{R^4}\, d\varphi\, d\theta =$$

$$\frac{3\pi}{128\rho^5}\Big\{4\rho(1-\rho)^2(1+\rho)^2 + \frac{4}{3}(1+3\rho^2)(3\rho+\rho^3) + 8\rho(3+2\rho^2+3\rho^4) -$$

$$8(1+\rho^2)(1-\rho^2)^2[\ln(1+\rho) - \ln(1-\rho)] - 16\rho(1+\rho^2)^2\Big\}$$

The function $I_2(\rho)$ also decreases as $\rho$ grows, consequently

$$I_2(\rho) \leqslant I_2(0) = H_2(0) = 4\pi/5$$

$$K = \int R^2_{,3} n_3^2 dS_1 \leqslant \left(\int R^4_{,3} dS_1\right)^{1/2} \left(\int n_3^4\, dS_1\right)^{1/2} \leqslant 4\pi/5$$

Now all is ready for the consideration of the example.

**Example.** Let $V$ be a unit sphere from a material with $\nu = 0.3$. The center of the sphere coincides with the origin, and $G$ is a crack in the diametral plane of the sphere. We assume that the crack $G$ is a circle of radius $\rho$ whose center coincides with the center of the sphere. If $\rho = 0.23$, then the quantity $\theta$ estimated from (4.1) does not exceed $0.865$ and the operator $\Omega$ is compressive. If $\rho = 0.25$, then we obtain the estimate $\theta \leqslant 1.286$ from the calculations cited, and it is impossible to assert on the basis of this estimate that $\Omega$ is a compressive operator.

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